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q-Phase Operators of Finite Dimensional Two-Mode q-Oscillator Algebra

W.-S. Chung[1]

The q-phase operators are constructed for two-mode q-oscillators in a finite dimensional Hilbert space. It is shown that the q-coherent states for two-mode q-oscillators are not minimum uncertainty states.

1. INTRODUCTION

Phase operator has been an important topic in quantized electromagnetic field theory. First Susskind and Glogower (1964) studied it, but their phase operator is not hermitian. About 10 years ago, Pegg and Barnett (1965) suggested a formalism where all operators act in a finite-dimensional Hilbert space and the phase operator is hermitian.

In the past few years, q-deformed oscillator algebra (Biedenharn, 1989; Macfarlane, 1989) was introduced in search of the representation of quantum algebra. Recently, the hermitian phase operator for the single-mode electromagnetic field was given in the q-deformed case (Yang and Yu, 1995).

The purpose of this paper is to extend the previous results to two-mode case. Two-mode extension of q-oscillator algebra is not trivial because it should possess the $su_q(2)$-covariance. Thus, two-mode oscillator algebra is called $su_q(2)$-covariant oscillator algebra. Two-mode (or multimode) q-oscillator algebra was firstly introduced by Pusz and Woronowicz (1989).

2. HERMITIAN PHASE OPERATOR

$su_q(2)$-covariant oscillator algbera means two-mode oscillator algebra which is covariant under the $su_q(2)$ algebra. It is defined as

[1] Department of Physics and Research Institute of Natural Science, Gyeongsang National University, Jinju, 660-701, Korea.

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$$
\begin{aligned}
a_1 a_2 &= q a_2 a_1, \\
a_1^\dagger a_2^\dagger &= q^{-1} a_2^\dagger a_1^\dagger, \\
a_1 a_2^\dagger &= q a_2^\dagger a_1, \\
a_2 a_1^\dagger &= q a_1^\dagger a_2, \\
a_1 a_1^\dagger &= 1 + q^2 a_1^\dagger a_1, \\
a_2 a_2^\dagger &= 1 + q^2 a_2^\dagger a_2 + (q^2 - 1) a_1^\dagger a_1, \\
[N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, &\qquad [N_i, a_j] = -\delta_{ij} a_j, \quad (i, j = 1, 2),
\end{aligned}
\tag{1}
$$

where deformation parameter q is assumed to be real. In this case, $a_i^\dagger$ is an hermitian conjugate operator of a_i and the number operators N_i is hermitian. Let us consider the finite (but arbitrary large) dimensional state space V of $su_q(2)$-covariant oscillator algebra. Let $(s+1)^2$ be the dimension of V and s is some positive integer. The number state $|n, m\rangle \in V$ are assumed to be orthonormal:

$$
\begin{aligned}
\langle n', m' \mid n, m\rangle &= \delta_{nn'} \delta_{mm'}, \\
\sum_{n,m=0}^{s} |n, m\rangle\langle n, m| &= 1.
\end{aligned}
\tag{2}
$$

We require that the usual $su_q(2)$-covariant oscillator algebra in the infinite dimensional Hilbert space such as the q-annihilation, creation, and number operators, have corresponding operators of $su_q(2)$-covariant oscillator algebra in the finite dimensional spave V, which goes to the infinite dimensional space as s tends to infinity. These may be accomplished by the following definition

$$
\begin{aligned}
a_1^\dagger &= \sum_{n,m=1}^{s} \sqrt{[n]}\, |n, m\rangle\langle n-1, m\rangle, \\
a_2^\dagger &= \sum_{n,m=1}^{s} q^n \sqrt{[m]}\, |n, m\rangle\langle n, m-1\rangle, \\
a_1 &= \sum_{n,m=1}^{s} \sqrt{[n]}\, |n-1, m\rangle\langle n, m\rangle, \\
a_2 &= \sum_{n,m=1}^{s} q^n \sqrt{[m]}\, |n, m-1\rangle\langle n, m\rangle, \\
N_1 &= \sum_{n,m=1}^{s} n |n, m\rangle\langle n, m\rangle, \\
N_2 &= \sum_{n,m=1}^{s} m |n, m\rangle\langle n, m\rangle,
\end{aligned}
\tag{3}
$$

where q-number $[x]$ is defined as

$$[x] = \frac{q^{2x} - 1}{q^2 - 1}.$$

Applying $a_1^\dagger$, $a_2^\dagger$, a_1, and a_2 to the number eigenstate $|n, m\rangle$ gives

$$\begin{aligned}
a_1^\dagger |n, m\rangle &= \sqrt{[n+1]}\, |n+1, m\rangle, \quad (n = 0, 1, \ldots, s-1,\ m = 0, 1, \ldots, s) \\
a_1^\dagger |s, m\rangle &= 0, \quad (m = 0, 1, \ldots, s) \\
a_2^\dagger |n, m\rangle &= q^n \sqrt{[m+1]}\, |n, m+1\rangle, \quad (n = 0, 1, \ldots, s,\ m = 0, 1, \ldots, s-1) \\
a_2^\dagger |n, s\rangle &= 0, \quad (n = 0, 1, \ldots, s) \\
a_1 |n, m\rangle &= \sqrt{[n]}\, |n-1, m\rangle, \quad (n, m = 0, 1, \ldots, s) \\
a_2 |n, m\rangle &= q^n \sqrt{[m]}\, |n, m-1\rangle, \quad (n, m = 0, 1, \ldots, s)
\end{aligned} \tag{4}$$

then the commutation relation for creation and annihilation operators of the finite dimensional $su_q(2)$-covariant oscillator algebra becomes

$$\begin{aligned}
a_1 a_1^\dagger - q^2 a_1^\dagger a_1 &= 1 - [s+1] \sum_{m=0}^{s} |sm\rangle\langle sm|, \\
a_2 a_2^\dagger - q^2 a_2^\dagger a_2 &= 1 + (q^2 - 1) a_1^\dagger a_1 - [s+1] \sum_{n=0}^{s} q^{2n} |ns\rangle\langle ns|.
\end{aligned} \tag{5}$$

The polar decomposition for the operators (O) is an analogue of the complex number decomposition $z = re^{i\theta}$ and is defined as $O = UH$, where U is a partial isometry (unitary) operator and H a hermitian operator. Using the similar method in Pegg and Barnett (1989), one can obtain

$$\begin{aligned}
a_1 &= e^{i\Phi_1} \sqrt{[N_1]}, \\
a_1^\dagger &= \sqrt{[N_1]}\, e^{-i\Phi_1}, \\
a_2 &= e^{i\Phi_2} q^{N_1} \sqrt{[N_2]}, \\
a_2^\dagger &= \sqrt{[N_2]} q^{N_1}\, e^{-i\phi_2}.
\end{aligned} \tag{6}$$

From the relation (4), we have

$$\begin{aligned}
e^{i\Phi_1} |n, m\rangle &= |n-1, m\rangle, \\
\langle n, m| e^{-i\Phi_1} &= \langle n-1, m|, \quad (n = 1, \ldots, s,\ m = 0, \ldots, s) \\
e^{i\Phi_2} |n, m\rangle &= |n, m-1\rangle, \\
\langle n, m| e^{-i\Phi_2} &= \langle n, m-1|, \quad (n = 0, \ldots, s,\ m = 1, \ldots, s)
\end{aligned} \tag{7}$$

and

$$e^{i\Phi_1}|0, m\rangle = \sum_{k=0}^{s} C_k^{(1)}|k, m\rangle,$$
$$e^{i\Phi_2}|n, 0\rangle = \sum_{k=0}^{s} C_k^{(2)}|n, k\rangle. \tag{8}$$

From the Eqs. (7) and (8), we have

$$C_k^{(1)} = C_k^{(2)} = 0, \quad (k = 0, 1, \ldots, s-1) \tag{9}$$

and

$$e^{i\Phi_1}|0, m\rangle = C_s^{(1)}|s, m\rangle,$$
$$e^{i\Phi_2}|n, 0\rangle = C_s^{(2)}|n, s\rangle.$$

From the unitary requirement for $e^{i\Phi_1}$ and $e^{i\Phi_2}$, we can set

$$C_s^{(1)} = \psi_1, \qquad C_s^{(1)} = \psi_2, \tag{10}$$

where ψ_1 and ψ_2 are real numbers. The unitary phase operators $e^{i\Phi_1}$ and $e^{i\Phi_2}$ can then be written as projection operators

$$e^{i\Phi_1} = \sum_{n=1}^{s}\sum_{m=0}^{s} |n-1, m\rangle\langle n, m| + e^{i\psi_1}\sum_{m=0}^{s} |sm\rangle\langle 0m|,$$
$$e^{i\Phi_2} = \sum_{n=0}^{s}\sum_{m=1}^{s} |n, m-1\rangle\langle n, m| + e^{i\psi_2}\sum_{n=0}^{s} |ns\rangle\langle n0|. \tag{11}$$

The exponential phase operators are mutually commutative,

$$e^{i\Phi_1}\, e^{i\Phi_2} = e^{i\Phi_2}\, e^{i\Phi_1}. \tag{12}$$

Note that the Eq. (11) is q-independent and is the same as in in Pegg and Barnett (1989). We can say that the phase operator is not deformed in the q-deformed case. So the properties of $e^{i\Phi_1}$ and $e^{i\Phi_2}$ are the same with the undeformed one. It is worth noting that

$$a_1 = e^{i\Phi_1}\sqrt{[N_1]} \neq \sqrt{[N_1+1]}\, e^{i\Phi_1},$$
$$a_2 = e^{i\Phi_2} q^{N_1}\sqrt{[N_2]} \neq \sqrt{[N_2+1]}q^{N_1}\, e^{i\Phi_2} \tag{13}$$

in the finite dimensional case, as is different from the infinite dimensional case. Instead, we have

$$a_1 = e^{i\Phi_1}\sqrt{[N_1]} = \sqrt{[N_1+1]}\, e^{i\Phi_1} - e^{i\psi_1}\sqrt{[s+1]}\sum_{m=0}^{s} |sm\rangle\langle 0m|,$$
$$a_2 = e^{i\Phi_2} q^{N_1}\sqrt{[N_2]} = \sqrt{[N_2+1]}q^{N_1}\, e^{i\Phi_2} - e^{i\psi_2}\sqrt{[s+1]}\sum_{n=0}^{s} |ns\rangle\langle n0|. \tag{14}$$

Thus the properties of $e^{i\Phi_1}$ and $e^{i\Phi_2}$ are the same with the undeformed one. Let the eigenstate of $e^{i\Phi_1}$ and $e^{i\Phi_2}$ be

$$|\theta_n, \theta_m\rangle = \frac{1}{s+1} \sum_{n,m=0}^{s} e^{i(\theta_n n + \theta_m m)} |n, m\rangle \tag{15}$$

which obeys the eigenequation

$$e^{i\Phi_i} |\theta_n, \theta_m\rangle = e^{i\phi_i} |\theta_n, \theta_m\rangle \tag{16}$$

with

$$e^{i\phi_1} = e^{i\frac{\psi_1 + 2n\pi}{s+1}}$$
$$e^{i\phi_2} = e^{i\frac{\psi_2 + 2m\pi}{s+1}}$$

and

$$\theta_n = \theta_0 + \frac{2n\pi}{s+1}$$
$$\theta_m = \theta_0 + \frac{2m\pi}{s+1}, \quad (0 \le n,\ m \le s)$$
$$\theta_0 = \frac{\psi_1}{s+1} = \frac{\psi_2}{s+1}$$

where the value of θ_0 are arbitrary and $\psi_1 = \psi_2$ is assumed for simplicity.

3. *q*-DEFORMED OPERATORS OF PHASE QUANTA

From Eq. (7), we know that the operators $e^{i\Phi_1}$ and $e^{i\Phi_2}$ play the roles of step operators. And the operators $e^{-\frac{2\pi i}{s+1}N_1}$ and $e^{-\frac{2\pi i}{s+1}N_2}$ are also step operators with repect to the phase states $|\theta_n, \theta_m\rangle$, because

$$\begin{aligned}
e^{-\frac{2\pi i}{s+1}N_1}|\theta_n, \theta_m\rangle &= |\theta_{n-1}, \theta_m\rangle, \quad (n \neq 0)\\
e^{-\frac{2\pi i}{s+1}N_1}|\theta_0, \theta_m\rangle &= |\theta_s, \theta_m\rangle,\\
e^{\frac{2\pi i}{s+1}N_1}|\theta_n, \theta_m\rangle &= |\theta_{n+1}, \theta_m\rangle, \quad (n \neq s)\\
e^{\frac{2\pi i}{s+1}N_1}|\theta_s, \theta_m\rangle &= |\theta_0, \theta_m\rangle,\\
e^{-\frac{2\pi i}{s+1}N_2}|\theta_n, \theta_m\rangle &= |\theta_n, \theta_{m-1}\rangle, \quad (m \neq 0)\\
e^{-\frac{2\pi i}{s+1}N_2}|\theta_n, \theta_0\rangle &= |\theta_n, \theta_s\rangle,\\
e^{\frac{2\pi i}{s+1}N_2}|\theta_n, \theta_m\rangle &= |\theta_n, \theta_{m+1}\rangle, \quad (m \neq s)\\
e^{\frac{2\pi i}{s+1}N_2}|\theta_n, \theta_s\rangle &= |\theta_n, \theta_0\rangle.
\end{aligned} \tag{17}$$

Using Eq. (19), $e^{-\frac{2\pi i}{s+1}N_1}$ and $e^{-\frac{2\pi i}{s+1}N_2}$ can be written in terms of the phase states;

$$\begin{aligned} e^{-\frac{2\pi i}{s+1}N_1} &= \sum_{n=1}^{s}\sum_{m=0}^{s}|\theta_{n-1},\theta_m\rangle\langle\theta_n,\theta_m| + \sum_{m=0}^{s}|\theta_s,\theta_m\rangle\langle\theta_0\theta_m|,\\ e^{-\frac{2\pi i}{s+1}N_2} &= \sum_{n=0}^{s}\sum_{m=1}^{s}|\theta_n,\theta_{m-1}\rangle\langle\theta_n,\theta_m| + \sum_{n=0}^{s}|\theta_n,\theta_s\rangle\langle\theta_n\theta_0|. \end{aligned} \tag{18}$$

Thus, we can also define the q-deformed annihilation operators of phase quanta as follows;

$$\begin{aligned} \sigma_1 &= e^{-\frac{2\pi i}{s+1}N_1}\sqrt{[\Phi_1]},\\ \sigma_2 &= e^{-\frac{2\pi i}{s+1}N_2}e^{\Phi_1}\sqrt{[\Phi_2]} \end{aligned} \tag{19}$$

and $\sigma_i^\dagger$ is obtained by taking the hermitian conjugation in the expression for σ_i. Then, the phase operators read

$$\begin{aligned} \Phi_1 &= \sum_{n,m=0}^{s}\theta_n|\theta_n,\theta_m\rangle\langle\theta_n,\theta_m|,\\ \Phi_2 &= \sum_{n,m=0}^{s}\theta_m|\theta_n,\theta_m\rangle\langle\theta_n,\theta_m|. \end{aligned} \tag{20}$$

The q-annihilation operators σ_i's satisfy

$$\begin{aligned} \sigma_1|\theta_n,\theta_m\rangle &= \sqrt{[\theta_n]}|\theta_{n-1},\theta_m\rangle, \quad (n\neq 0)\\ \sigma_1|\theta_0,\theta_m\rangle &= \sqrt{[\theta_0]}|\theta_s,\theta_m\rangle,\\ \sigma_2|\theta_n,\theta_m\rangle &= q^{\theta_n}\sqrt{[\theta_m]}|\theta_n,\theta_{m-1}\rangle, \quad (m\neq s)\\ \sigma_2|\theta_n,\theta_0\rangle &= q^{\theta_n}\sqrt{[\theta_0]}|\theta_n,\theta_s\rangle,\\ \sigma_1^\dagger|\theta_n,\theta_m\rangle &= \sqrt{[\theta_{n+1}]}|\theta_{n+1},\theta_m\rangle, \quad (n\neq s)\\ \sigma_1^\dagger|\theta_s,\theta_m\rangle &= \sqrt{[\theta_0]}|\theta_0,\theta_m\rangle,\\ \sigma_2^\dagger|\theta_n,\theta_m\rangle &= q^{\theta_n}\sqrt{[\theta_{m+1}]}|\theta_n,\theta_{m+1}\rangle, \quad (m\neq s)\\ \sigma_2^\dagger|\theta_m,\theta_s\rangle &= q^{\theta_n}\sqrt{[\theta_0]}|\theta_n,\theta_0\rangle, \end{aligned} \tag{21}$$

That is to say that σ_i and $\sigma_i^\dagger$ operators can be written as

$$\begin{aligned} \sigma_1 &= \sum_{n=1}^{s}\sum_{m=0}^{s}\sqrt{[\theta_n]}|\theta_{n-1},\theta_m\rangle\langle\theta_n,\theta_m| + \sum_{m=0}^{s}\sqrt{[\theta_0]}|\theta_s,\theta_m\rangle\langle\theta_0,\theta_m|,\\ \sigma_2 &= \sum_{n=0}^{s}\sum_{m=1}^{s}q^{\theta_n}\sqrt{[\theta_m]}|\theta_n,\theta_{m-1}\rangle\langle\theta_n,\theta_m| + \sum_{n=0}^{s}q^{\theta_n}\sqrt{[\theta_0]}|\theta_n,\theta_s\rangle\langle\theta_n,\theta_0|, \end{aligned}$$

$$\sigma_1^\dagger = \sum_{n=1}^{s}\sum_{m=0}^{s}\sqrt{[\theta_n]}|\theta_n,\theta_m\rangle\langle\theta_{n-1},\theta_m| + \sum_{m=0}^{s}\sqrt{[\theta_0]}|\theta_0,\theta_m\rangle\langle\theta_s,\theta_m|,$$

$$\sigma_2^\dagger = \sum_{n=0}^{s}\sum_{m=1}^{s} q^{\theta_n}\sqrt{[\theta_m]}|\theta_n,\theta_m\rangle\langle\theta_n,\theta_{m-1}| + \sum_{n=0}^{s} q^{\theta_n}\sqrt{[\theta_0]}|\theta_n,\theta_0\rangle\langle\theta_n,\theta_s| \quad (22)$$

and their commutation relations are

$$\sigma_1\sigma_1^\dagger - q^{\frac{4\pi}{s+1}}\sigma_1^\dagger\sigma_1 = \left[\frac{2\pi}{s+1}\right] + ([\theta_0]-[\theta_{s+1}])\sum_{m=0}^{s}|\theta_s,\theta_m\rangle\langle\theta_s,\theta_m|,$$

$$\sigma_2\sigma_2^\dagger - q^{\frac{4\pi}{s+1}}\sigma_2^\dagger\sigma_2 = \left[\frac{2\pi}{s+1}\right](1+(q^2-1)\sigma_1^\dagger\sigma_1) \quad (23)$$

$$+([\theta_0]-[\theta_{s+1}])\sum_{n=0}^{s} q^{2\theta_n}|\theta_n,\theta_s\rangle\langle\theta_n,\theta_s|,$$

where we used the following identity

$$[\theta_{n+1}] - q^{\frac{4\pi}{s+1}}[\theta_n] = \left[\frac{2\pi}{s+1}\right].$$

The remaining commutation relations are given by

$$\sigma_1\sigma_2 = q^{\frac{2\pi}{s+1}}\sigma_2\sigma_1 + \sum_{m=0}^{s-1}\left(q^{\theta_0} - q^{\theta_s+\frac{2\pi}{s+1}}\right)\sqrt{[\theta_0][\theta_{m+1}]}|\theta_s,\theta_m\rangle\langle\theta_0,\theta_{m+1}|$$

$$+[\theta_0]\left(q^{\theta_0} - q^{\theta_s+\frac{2\pi}{s+1}}\right)|\theta_s,\theta_s\rangle\langle\theta_0,\theta_0|, \quad (24)$$

$$\sigma_1\sigma_2^\dagger = q^{\frac{2\pi}{s+1}}\sigma_2^\dagger\sigma_1 + \sum_{m=1}^{s}\left(q^{\theta_0} - q^{\theta_s+\frac{2\pi}{s+1}}\right)\sqrt{[\theta_0][\theta_n]}|\theta_{n-1},\theta_0\rangle\langle\theta_n,\theta_s|$$

$$+[\theta_0]\left(q^{\theta_0} - q^{\theta_s+\frac{2\pi}{s+1}}\right)|\theta_s,\theta_0\rangle\langle\theta_0,\theta_s|.$$

The commutation relations between phase operators and q-annihilation operators for phase quanta are

$$[\Phi_1,\sigma_1] = -\frac{2\pi}{s+1}\sigma_1 + 2\pi\sum_{m=0}^{s}\sqrt{[\theta_0]}|\theta_s,\theta_m\rangle\langle\theta_0,\theta_m|,$$

$$[\Phi_2,\sigma_2] = -\frac{2\pi}{s+1}\sigma_2 + 2\pi\sum_{n=0}^{s} q^{\theta_n}\sqrt{[\theta_0]}|\theta_n,\theta_s\rangle\langle\theta_n,\theta_0|, \quad (25)$$

$$[\Phi_1,\sigma_2] = [\Phi_2,\sigma_1] = 0.$$

From the above formula, we obtain the following expression

$$(\sigma_1^\dagger)^k(\sigma_2^\dagger)^l|\theta_0,\theta_0\rangle = q^{l\theta_0}\left(\prod_{p=1}^{k}[\theta_p]\prod_{j=1}^{l}[\theta_j]\right)^{1/2}|\theta_k,\theta_l\rangle,\quad (1\le k,\ l\le s)$$

$$(\sigma_1^\dagger)^k(\sigma_2^\dagger)^l|\theta_0,\theta_0\rangle = q^{l\theta_0}\left(\prod_{i=0}^{s}[\theta_i]\right)^{\frac{n+n'}{2}}\left(\prod_{p=1}^{m}[\theta_p]\prod_{j=1}^{m'}[\theta_j]\right)^{1/2}|\theta_m,\theta_{m'}\rangle, \qquad (26)$$

$$(k=(s+1)n+m,\ l=(s+1)n'+m')$$

where the value of θ_0 are arbitrary.

4. *q*-COHERENT STATES

We now define a new unnormalized q-coherent states of the finite dimensional two-mode oscillators as follows;

$$|z_1,z_2\rangle = \sum_{n,m=0}^{s}\frac{z_1^n z_2^m}{\sqrt{[n]![m]!}}|n,m\rangle. \qquad (27)$$

The norm of the q-coherent state is easily computed

$$\langle z_1,z_2\mid z_1,z_2\rangle = e_s(x_1)e_s(x_2),$$

where finite dimensional q-exponential function $e_s(x)$ is defined as

$$e_s(x) = \sum_{k=0}^{s}\frac{x^k}{[k]!}, \qquad (28)$$

and z_1 and z_2 are taken to be ordinary (commuting) complex variables. Obviously, when $s\to\infty$, $e_s(x)$ becomes infinite dimensional q-exponential function and $|z_1,z_2\rangle$ becomes q-coherent state of two-mode q-oscillators in an infinite-dimensional Hilbert space. Acting the q-annihilation operators on the q-coherent states gives

$$a_1|z_1,z_2\rangle = z_1\left(|z_1,z_2\rangle - \frac{z_1^s}{\sqrt{[s]!}}\sum_{m=0}^{s}\frac{z_2^m}{\sqrt{[m]!}}|sm\rangle\right),$$

$$a_2|z_1,z_2\rangle = z_2\left(|qz_1,z_2\rangle - \frac{z_2^s}{\sqrt{[s]!}}\sum_{n=0}^{s}\frac{q^n z_1^m}{\sqrt{[n]!}}|ns\rangle\right), \qquad (29)$$

which implies that the q-coherent state defined in Eq. (29) is not an eigenstate of the annihilation operator. The second terms on the right hand side of Eq. (31) come from the finiteness of the dimension. The q-coherent state obeys

$$a_1a_2|z_1,z_2\rangle = qa_2a_1|z_1,z_2\rangle,$$

which is consistent with the commutation relations of the algebra (1).

Now we introduce the q-position and q-momentum operators X_i and P_i as follows;

$$X_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger), \qquad P_i = \sqrt{\frac{m\hbar\omega}{2}}(a_i^\dagger - a_i), \tag{30}$$

where $i = 1, 2$ and they satisfy

$$[X_i, P_i] = i\hbar[a_i, a_i^\dagger]. \tag{31}$$

The uncertainty relation for q-position operator and q-momentum operator is given by

$$\langle(\Delta X_i)^2\rangle\langle(\Delta P_i)^2\rangle \geq \frac{1}{4}|\langle[X_i, P_i]\rangle|^2, \tag{32}$$

where $|z\rangle = |z_1, z_2\rangle$ and

$$\langle A\rangle = \frac{\langle z|A|z\rangle}{\langle z \mid z\rangle}.$$

It is well known that the coherent states of an ordinary harmonic oscillator minimize the uncertainty relation of X_i and P_i. Now, we have a question: does the q-coherent state defined in Eq. (29) minimize the uncertainty relation for q-position and q-momentum operators. In order to check this, we should compute some expectation values for step operators and their combinations with respect to the q-coherent state.

For the q-coherent state, we have

$$\langle a_1\rangle = z_1\frac{e_{s-1}(x_1)}{e_s(x_1)},$$

$$\langle a_2\rangle = z_2\frac{e_s(qx_1)e_{s-1}(x_2)}{e_s(x_1)e_s(x_2)},$$

$$\langle a_i^\dagger\rangle = \langle a_i\rangle^*,$$

$$\langle a_1^2\rangle = z_1^2\frac{e_{s-2}(x_1)}{e_s(x_1)},$$

$$\langle a_2^2\rangle = z_2^2\frac{e_s(q^2x_1)e_{s-2}(x_2)}{e_s(x_1)e_s(x_2)},$$

$$\langle a_i^{\dagger 2}\rangle = \langle a_i^2\rangle^*,$$

$$\langle a_1^\dagger a_1\rangle = x_1\frac{e_{s-1}(x_1)}{e_s(x_1)},$$

$$\langle a_2^\dagger a_2 \rangle = x_2 \frac{e_s(q^2 x_1) e_{s-1}(x_2)}{e_s(x_1) e_s(x_2)},$$

$$\langle a_1 a_1^\dagger \rangle = \frac{e_{s-1}(x_1) + q^2 x_1 e_{s-2}(x_1)}{e_s(x_1)},$$

$$\langle a_2 a_2^\dagger \rangle = \frac{e_s(q^2 x_1)(e_{s-1}(x_2) + q^2 x_2 e_{s-2}(x_2))}{e_s(x_1) e_s(x_2)},$$

$$= \frac{e_s(q^2 x_1)(e_{s-1}(q^2 x_2) + x_2 e_{s-2}(x_2))}{e_s(x_1) e_s(x_2)},$$

where we used

$$\langle z \mid ns \rangle = \frac{(z_1^*)^n (z_2^*)^s}{\sqrt{[n]![s]!}},$$

$$\langle z \mid sm \rangle = \frac{(z_1^*)^s (z_2^*)^m}{\sqrt{[s]![m]!}},$$

$$e_s(x) + (q^2 - 1) x e_{s-1}(x) = e_s(q^2 x)$$

and x_i implies that $x_i = z_i^* z_i = |z_i|^2$. So, we obtain

$$\langle X_1 \rangle = \sqrt{\frac{\hbar}{2m\omega}} (z_1^* + z_1) \frac{e_{s-1}(x_1)}{e_s(x_1)},$$

$$\langle X_2 \rangle = \sqrt{\frac{\hbar}{2m\omega}} (z_2^* + z_2) \frac{e_s(q x_1) e_{s-1}(x_2)}{e_s(x_1) e_s(x_2)},$$

$$\langle P_1 \rangle = i \sqrt{\frac{m\hbar\omega}{2}} (z_1^* - z_1) \frac{e_{s-1}(x_1)}{e_s(x_1)},$$

$$\langle P_2 \rangle = \sqrt{\frac{m\hbar\omega}{2}} (z_2^* - z_2) \frac{e_s(q x_1) e_{s-1}(x_2)}{e_s(x_1) e_s(x_2)},$$

$$\left\langle X_1^2 \right\rangle = \frac{\hbar}{2m\omega} \frac{\left(z_1^2 + (z_1^*)^2 + q^2 x_1\right) e_{s-2}(x_1) + (x_1 + 1) e_{s-1}(x_1)}{e_s(x_1)},$$

$$\left\langle X_2^2 \right\rangle = \frac{\hbar}{2m\omega} \frac{e_s(q^2 x_1)\left[\left(z_2^2 + (z_2^*)^2 + q^2 x_2\right) e_{s-2}(x_2) + (x_2 + 1) e_{s-1}(x_2)\right]}{e_s(x_1) e_s(x_2)},$$

$$\left\langle P_1^2 \right\rangle = -\frac{m\hbar\omega}{2} \frac{\left(z_2^2 + (z_1^*)^2 - q^2 x_1\right) e_{s-2}(x_1) + (-x_1 - 1) e_{s-1}(x_1)}{e_s(x_1)},$$

$$\left\langle P_2^2\right\rangle = -\frac{m\hbar\omega}{2}\frac{e_s(q^2x_1)\left[\left(z_2^2+(z_2^*)^2-q^2x_2\right)e_{s-2}(x_2)+(-x_2-1)e_{s-1}(x_2)\right]}{e_s(x_1)e_s(x_2)},$$

$$\Delta X_1^2 = \frac{\hbar}{2m\omega}[e_s(x_1)]^{-2}\left[\left(z_1^2+(z_1^*)^2+q^2x_1\right)e_s(x_1)e_{s-2}(x_1)\right.$$
$$\left.+(x_1+1)e_{s-1}(x_1)e_s(x_1)-(z_1+z_1^*)^2(e_{s-1}(x_1))^2\right],$$

$$\Delta X_2^2 = \frac{\hbar}{2m\omega}[e_s(x_1)e_s(x_2)]^{-2}\left[\left(z_2^2+(z_2^*)^2+q^2x_2\right)e_s(x_1)e_s(q^2x_1)\right.$$
$$\times e_s(x_2)e_{s-2}(x_2)+(x_2+1)e_s(x_1)e_s(q^2x_1)e_s(x_2)e_{s-1}(x_2)$$
$$\left.-(z_2+z_2^*)^2(e_s(qx_1)e_{s-1}(x_2))^2\right],$$

$$\Delta P_1^2 = -\frac{m\hbar\omega}{2}[e_s(x_1)]^{-2}\left[\left(z_1^2+(z_1^*)^2-q^2x_1\right)e_s(x_1)e_{s-2}(x_1)\right.$$
$$\left.-(x_1+1)e_{s-1}(x_1)e_s(x_1)-(z_1^*-z_1)^2(e_{s-1}(x_1))^2\right],$$

$$\Delta P_2^2 = -\frac{m\hbar\omega}{2}[e_s(x_1)e_s(x_2)]^{-2}\left[\left(z_2^2+(z_2^*)^2-q^2x_2\right)e_s(x_1)e_s(q^2x_1)\right.$$
$$\times e_s(x_2)e_{s-2}(x_2)-(x_2+1)e_s(x_1)e_s(q^2x_1)e_s(x_2)e_{s-1}(x_2)$$
$$\left.-(z_2^*-z_2)^2(e_s(qx_1)e_{s-1}(x_2))^2\right],$$

$$\langle[X_1, P_1]\rangle = i\hbar\frac{(1-x_1)e_{s-1}(x_1)+q^2x_1e_{s-2}(x_1)}{e_s(x_1)},$$

$$\langle[X_2, P_2]\rangle = i\hbar\frac{e_s(q^2x_1)((1-x_2)e_{s-1}(x_2)+q^2x_2e_{s-2}(x_2))}{e_s(x_1)e_s(x_2)}. \tag{33}$$

Equation (34) shows that the q-coherent states in finite dimensional Hilbert space are not minimum uncertainty states.

5. CONCLUSION

In this paper, I have studied the q-deformed phase operator for two-mode q-oscillators. I introduced the finite dimensional two-mode q-oscillator algebra which has $su_q(2)$-covariance and expressed all operators in terms of number states and phase states. Using the previous results, I constructed the hermitian q-phase operators for two-mode q-oscillators. Finally, I obtained the q-coherent state explicitly. As is different from the undeformed case, it was shown that the q-coherent states are not coherent states in the ordinary sense because they are neither minimum uncertainty states nor eigenstates of q-annihilation operators.

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REFERENCES

Biedenharn, L. (1989). *Journal of Physics A* **22**, L873.
Macfarlane, A. (1989). *Journal of Physics A* **22**, 4581.
Pegg, D. and Barnett, S. (1989). *Physical Review D* **39**, 1665.
Pusz, W. and Woronowicz, S. (1989). *Reports on Mathematical Physics* **27**, 231.
Susskind, L. and Glogower, J. (1964). *Physics* **1**, 49.
Yang, Y. and Yu, Z. (1995). *Modern Physics Letters A* **10**, 347.